数理解析研究所講究録
第 1686 巻 2010 年 63-72

Remarks on the preservation of topological covering properties under Cohen forcing

Masaru Kada*† / 嘉田 勝
Osaka Prefecture University / 大阪府立大学

Abstract

Iwasa investigated the preservation of various covering properties of topological spaces under Cohen forcing. By improving the argument in Iwasa's paper, we prove that the Rothberger property, the Menger property and selective screenability are also preserved under Cohen forcing and forcing with the measure algebra.

1 Introduction

Let (X, τ) be a topological space and $\mathbb{P}$ a forcing notion. In the forcing extension by $\mathbb{P}$, the collection τ of subsets of X is no longer a topology on X, since there are more infinite subsets of τ, whose union may not belong to τ. However, τ is still a base of a topology on X. So we let $\check{\tau}^{\mathbb{P}}$ denote a $\mathbb{P}$-name for a topology on $\check{X}$ which is generated by τ, and we consider $(\check{X}, \tau^{\mathbb{P}})$ as a topological space corresponding to (X, τ) in the forcing extension.

For a property Φ of a topological space, We say a forcing notion $\mathbb{P}$ *preserves* Φ if, whenever (X, τ) satisfies Φ, we have $\Vdash_{\mathbb{P}}$ "$(\check{X}, \tau^{\mathbb{P}})$ satisfies Φ".

Grunberg, Junqueira and Tall [4] proved that Cohen forcing preserves paracompactness. Using their ideas, Iwasa [5] extensively studied the preservation of various covering properties of topological spaces under Cohen extensions, and proved that Cohen forcing preserves the following properties: paracompactness, subparacompactness, screenability, σ-metacompactness, σ-paraLindelöfness, Lindelöfness and meta-Lindelöfness. It is not so hard to observe, though it is not explicitly stated, that we can prove the same preservation results also for the measure algebra.

* Supported by Grant-in-Aid for Young Scientists (B) 21740080, MEXT.
† email: kada@mi.s.osakafu-u.ac.jp

In the present paper, we will improve the idea used in Iwasa's paper and prove that the following properties are also preserved under Cohen forcing and forcing with the measure algebra:

(1) the Rothberger property,
(2) the Menger property,
(3) selective screenability.

A space (X,τ) *has the Rothberger property* if, for every sequence $\langle \mathcal{U}_n : n < \omega \rangle$ of open covers of (X,τ) there is a sequence $\langle U_n : n < \omega \rangle$ of open sets of (X,τ) such that

- for all $n < \omega$, $U_n \in \mathcal{U}_n$, and
- $\{U_n : n < \omega\}$ is an open cover of (X,τ).

A space (X,τ) *has the Menger property* if, for every sequence $\langle \mathcal{U}_n : n < \omega \rangle$ of open covers of (X,τ) there is a sequence $\langle \mathcal{F}_n : n < \omega \rangle$ of sets of open sets of (X,τ) such that

- for all $n < \omega$, $\mathcal{F}_n$ is a finite subset of $\mathcal{U}_n$, and
- $\bigcup_{n<\omega} \mathcal{F}_n$ is an open cover of (X,τ).

It is easy to see that the Rothberger property implies the Menger property, and the Menger property implies Lindelöfness.

For a topological space (X,τ) and two sets $\mathcal{A},\mathcal{B}$ of open sets of (X,τ), we say $\mathcal{A}$ *refines* $\mathcal{B}$, or $\mathcal{A}$ is a *refinement* of $\mathcal{B}$, if for each $U \in \mathcal{A}$ there is $V \in \mathcal{B}$ with $U \subseteq V$. We will use this terminology even if $\mathcal{A}$ or $\mathcal{B}$ is not a cover of (X,τ).

A space (X,τ) is *selectively screenable*[*1] if, for every sequence $\langle \mathcal{U}_n : n < \omega \rangle$ of open covers of (X,τ), there is a sequence $\langle \mathcal{H}_n : n < \omega \rangle$ of sets of open sets of (X,τ) such that

- for all $n < \omega$, $\mathcal{H}_n$ is pairwise disjoint and refines $\mathcal{U}_n$, and
- $\bigcup_{n<\omega} \mathcal{H}_n$ is an open cover of (X,τ).

It is easy to see that a space with the Rothberger property is selectively screenable.

For an infinite cardinal κ, let $\mathbb{C}(\kappa)$ denote the Cohen forcing notion with the index set κ (that is, $\mathbb{C}(\kappa) = \mathrm{Fn}(\kappa,2)$), and $\mathbb{B}(\kappa)$ denote the measure algebra on 2^κ.

Remark 1. Scheepers and Tall proved that, if (X,τ) is a Lindelöf space and $\kappa \geq \aleph_1$, then $\Vdash_{\mathbb{C}(\kappa)}$ "$(\check{X},\tau^{\mathbb{P}})$ has the Rothberger property" [7, Theorem 11]. Since the Rothberger property implies Lindelöfness, their result means that, for $\kappa \geq \aleph_1$, forcing with $\mathbb{C}(\kappa)$ preserves the Rothberger property. They also proved that, for any infinite

---

[*1] Selective screenability was introduced by Addis and Gresham [1] and called property C. The term "selectively screenable" was coined by Babinkostova in her papers, to avoid the confusion with strong measure zero, which was also called property C in the old days.

cardinal κ, forcing with $\mathbb{B}(\kappa)$ preserves the Rothberger property [7, Theorem 15]. Theorem 3.6 in the present paper fills the "missing part", that is, the preservation of the Rothberger property under forcing with $\mathbb{C}(\aleph_0)$. It is a still unsolved problem whether forcing with $\mathbb{C}(\aleph_0)$ converts a Lindelöf space in the ground model into a space with the Rothberger property.

Scheepers and Tall also proved that the Menger property is preserved under forcing with $\mathbb{B}(\kappa)$ for any infinite cardinal κ [7, Theorem 42], but it has not been stated that the Menger property is preserved under Cohen forcing. Theorem 3.8 in the present paper was pointed out by Scheepers.

2 Endowments and approximation of open covers

The combinatorial concept of *endowments*, also spelled *n-dowments*[*2], of the Cohen forcing notion $\mathbb{C}(\kappa)$ was originally invented by Dow [2], and used in [3] and [4] in connection with the preservation of topological properties under Cohen forcing.

Although endowments are originally defined only for the Cohen forcing notion, here we redefine endowments for a general forcing notion.

Definition 2.1. A forcing notion $\mathbb{P}$ is *endowed* if there are a decomposition of $\mathbb{P}$ into an increasing union of length ω, say $\mathbb{P} = \bigcup_{n<\omega} P_n$ where $P_n \subseteq P_{n+1}$ for all n, and a sequence $\langle \mathcal{L}_n : n < \omega \rangle$ of sets with the following properties: For each $n < \omega$,

(1) $\mathcal{L}_n$ is a set of finite antichains in $\mathbb{P}$,
(2) for every maximal antichain A in $\mathbb{P}$, there is $L \in \mathcal{L}_n$ with $L \subseteq A$, and
(3) for any $p \in P_n$ and any n elements $L_0, \dots, L_{n-1}$ of $\mathcal{L}_n$, there are $q_0, \dots, q_{n-1}$ such that, $q_i \in L_i$ for each $i < n$, and the set $\{p, q_0, \dots, q_{n-1}\}$ has a lower bound in $\mathbb{P}$.

We call a sequence $\langle \mathcal{L}_n : n < \omega \rangle$ which meets the above requirements a *sequence of endowments* of $\mathbb{P}$, and we will say $\mathbb{P}$ *is endowed with* $\langle \mathcal{L}_n : n < \omega \rangle$. We call each $\mathcal{L}_n$ an *endowment*, an *n-dowment*[*3] or an *n-th endowment*.

Cohen forcing notion $\mathbb{C}(\kappa)$ is typically decomposed into the increasing union $\mathbb{C}(\kappa) = \bigcup_{n<\omega} C_n$ where $C_n = \{p \in \mathbb{C}(\kappa) : |p| \leq n\}$, and it is actually endowed with respect to this decomposition. The following result is called "Dow's Lemma" [3, Lemma 1.1].

[*2] The term "n-dowment" was coined, probably after Alan Dow, by one of the authors of the paper [3] other than Dow himself. Some other people, including Fleissner, have called the same structure a *lynx*.

[*3] The letter n in the term "n-dowment" may, *but does not have to*, be considered as a parameter, only when one set $\mathcal{L}_n$ for a specific natural number n is mentioned and the very specific alphabet "n" (not k, i, etc.) is chosen as a variable. Otherwise we should regard the letter n just as a part of the name and not for a variable.

Theorem 2.2. *For any infinite cardinal κ, the Cohen forcing notion $\mathbb{C}(\kappa)$ is endowed.*

However, the clause (3) in the definition of endowments is too strong for our purpose in the present paper. So we relax the clause (3) and define the notion of *weak endowments.*

Definition 2.3. A forcing notion $\mathbb{P}$ is *weakly endowed* if there are a decomposition of $\mathbb{P}$ into an increasing union of length ω, say $\mathbb{P} = \bigcup_{n<\omega} P_n$ where $P_n \subseteq P_{n+1}$ for all n, and a sequence $\langle \mathcal{L}_n : n < \omega \rangle$ of sets with the following properties: For each $n < \omega$,

(1) $\mathcal{L}_n$ is a set of finite antichains in $\mathbb{P}$,
(2) for every maximal antichain A in $\mathbb{P}$, there is $L \in \mathcal{L}_n$ with $L \subseteq A$, and
(3′) for any $p \in P_n$ and $L \in \mathcal{L}_n$, there is $q \in L$ such that p, q are compatible in $\mathbb{P}$.

We call a sequence $\langle \mathcal{L}_n : n < \omega \rangle$ which meets the above requirements a *sequence of weak endowments* of $\mathbb{P}$, and we will say $\mathbb{P}$ *is weakly endowed with* $\langle \mathcal{L}_n : n < \omega \rangle$. We call each $\mathcal{L}_n$ a *weak endowment* or an *n-th weak endowment.*

The following proposition is essentially proved in a sublemma [3, Lemma 1.0] for the proof of Dow's Lemma. For self-containedness, we will present a proof in Appendix.

Proposition 2.4. *For any infinite cardinal κ, the Cohen forcing notion $\mathbb{C}(\kappa)$ is weakly endowed.*

We can see that the measure algebra is also weakly endowed.

Theorem 2.5. *For any infinite cardinal κ, the measure algebra $\mathbb{B}(\kappa)$ is weakly endowed.*

Proof. Just decompose $\mathbb{B}(\kappa)$ into the increasing union $\mathbb{B}(\kappa) = \bigcup_{n<\omega} B_n$ where $B_n = \{p \in \mathbb{B}(\kappa) : \mu(p) \geq 2^{-n}\}$ and, for each n, let $\mathcal{L}_n$ be the collection of all finite antichains in $\mathbb{B}(\kappa)$ whose total measure is greater than $1 - 2^{-n}$. $\square$

Let (X, τ) be a topological space. Using endowments of $\mathbb{C}(\kappa)$, we can nicely approximate an open cover $\dot{\mathcal{U}}$ of $(\check{X}, \tau^{\mathbb{C}(\kappa)})$ by an open cover of (X, τ) in the ground model, which was the idea used in [3] and [4]. We review this idea, in a generalized representation for weakly endowed forcing notions.

Throughout the rest of the present paper, we assume that $\mathbb{P}$ is a weakly endowed forcing notion, and fix a corresponding decomposition $\mathbb{P} = \bigcup_{n<\omega} P_n$ and a sequence $\langle \mathcal{L}_n : n < \omega \rangle$ of weak endowments of $\mathbb{P}$.

Let $\dot{\mathcal{U}}$ be a $\mathbb{P}$-name for an open cover of $(\check{X}, \tau^{\mathbb{P}})$. For each $n < \omega$, we will construct an open cover $\mathcal{V}_n(\dot{\mathcal{U}})$ of (X, τ), which we will call the *n-th approximation of $\dot{\mathcal{U}}$ with respect to $\langle \mathcal{L}_n : n < \omega \rangle$.*

Since $\dot{\mathcal{U}}$ is forced to be an open cover of $(\check{X}, \tau^{\mathbb{P}})$ and τ is a base for $\tau^{\mathbb{P}}$, for each

$x \in X$ we can find a $\mathbb{P}$-name $\dot{W}_x$ for an element of τ such that

$$\Vdash_{\mathbb{P}} \text{“}\check{x} \in \dot{W}_x \text{ and } \exists U \in \dot{\mathcal{U}}\, (\dot{W}_x \subseteq U)\text{”}.$$

For each $x \in X$, choose a maximal antichain A_x in $\mathbb{P}$, and an open set $W_{x,p} \in \tau$ for each $p \in A_x$, so that $x \in W_{x,p}$ and $p \Vdash_{\mathbb{P}}$ “$\dot{W}_x = \check{W}_{x,p}$” for $p \in A_x$.

Now we fix $n < \omega$ and define $\mathcal{V}_n(\dot{\mathcal{U}})$ in the following way. For each $x \in X$, find $L_{x,n} \in \mathcal{L}_n$ such that $L_{x,n} \subseteq A_x$, and let $V_{x,n} = \bigcap\{W_{x,p} : p \in L_{x,n}\}$. Then $V_{x,n}$ is an open set containing x. Let $\mathcal{V}_n(\dot{\mathcal{U}}) = \{V_{x,n} : x \in X\}$.

The following property of $\mathcal{V}_n(\dot{\mathcal{U}})$ is easily observed.

Lemma 2.6. *For $n < \omega$, let $\mathcal{V}_n(\dot{\mathcal{U}})$ be the n-th approximation of $\dot{\mathcal{U}}$ with respect to $\langle \mathcal{L}_n : n < \omega\rangle$. Then for each $n < \omega$, for any $V \in \mathcal{V}_n(\dot{\mathcal{U}})$ and $p \in P_n$ there is $r \in \mathbb{P}$ such that $r \le p$ and $r \Vdash_{\mathbb{P}}$ “$\exists U \in \dot{\mathcal{U}}\, (\check{V} \subseteq U)$”.*

Proof. Fix $n < \omega$, $V \in \mathcal{V}_n(\dot{\mathcal{U}})$ and $p \in P_n$. Find $x \in X$ so that $V = V_{x,n}$ in the construction of $\mathcal{V}_n(\dot{\mathcal{U}})$, and look at $L_{x,n}$. By the property (3′) of the n-th weak endowment $\mathcal{L}_n$, find $q \in L_{x,n}$ and $r \in \mathbb{P}$ so that $r \le p$ and $r \le q$. By the definition of $V_{x,n}$, we have $V = V_{x,n} \subseteq W_{x,q}$. Since $\Vdash_{\mathbb{P}}$ “$\exists U \in \dot{\mathcal{U}}\, (\dot{W}_x \subseteq U)$” and $q \Vdash_{\mathbb{P}}$ “$\dot{W}_x = \check{W}_{x,q}$”, we have $r \Vdash_{\mathbb{P}}$ “$\exists U \in \dot{\mathcal{U}}\, (\check{V} \subseteq U)$”. □

3 Preservation of covering properties

Iwasa established the following result about the preservation of covering properties under Cohen forcing [5, Corollary 2.6]. Although he actually dealt only with Cohen forcing, the proof works for weakly endowed forcing notions.

Theorem 3.1. *The following covering properties are preserved under forcing with a weakly endowed forcing notion:*

(1) paracompactness,
(2) subparacompactness,
(3) screenability,
(4) σ-metacompactness,
(5) σ-paraLindelöfness,
(6) Lindelöfness,
(7) metaLindelöfness.

This section is devoted to the proof of the following preservation theorem.

Theorem 3.2. *The following covering properties are preserved under forcing with a weakly endowed forcing notion:*

(1) the Rothberger property,
(2) the Menger property,
(3) selective screenability.

The proof will be worked out by improving the idea used in Iwasa's paper. The following two lemmata are inspired by [5, Lemma 2.3].

Lemma 3.3. *Suppose that (X,τ) is a topological space, $\dot{\mathcal{U}}$ is a $\mathbb{P}$-name for an open cover of $(\check{X},\tau^{\mathbb{P}})$, $n<\omega$, and $\mathcal{H}\subseteq\tau$ is a refinement of $\mathcal{V}_n(\dot{\mathcal{U}})$. Then there is a $\mathbb{P}$-name $\dot{\mathcal{W}}$ for a subfamily of $\mathcal{H}$ with the following properties.*

(1) $\Vdash_{\mathbb{P}}$ "$\dot{\mathcal{W}}$ refines $\dot{\mathcal{U}}$".
(2) for any $p\in P_n$ and $H\in\mathcal{H}$ there is $r\in\mathbb{P}$ such that $r\leq p$ and $r\Vdash_{\mathbb{P}}$ "$\check{H}\in\dot{\mathcal{W}}$".

Proof. Construct a $\mathbb{P}$-name $\dot{\mathcal{W}}$ with the following properties: For each $H\in\mathcal{H}$, for $p\in\mathbb{P}$,

- if $p\Vdash_{\mathbb{P}}$ "$\exists U\in\dot{\mathcal{U}}\,(\check{H}\subseteq U)$", then $p\Vdash_{\mathbb{P}}$ "$\check{H}\in\dot{\mathcal{W}}$", and
- if $\forall r\leq p\,(r\nVdash_{\mathbb{P}}$ "$\exists U\in\dot{\mathcal{U}}\,(\check{H}\subseteq U)$") (equivalently, $p\Vdash_{\mathbb{P}}$ "$\forall U\in\dot{\mathcal{U}}\,(\check{H}\not\subseteq U)$"), then $p\Vdash_{\mathbb{P}}$ "$\check{H}\notin\dot{\mathcal{W}}$".

Such construction of $\dot{\mathcal{W}}$ can be done using the "maximal principle" (see [6, VII Theorem 8.2]). Use Lemma 2.6 to check that this $\dot{\mathcal{W}}$ is as desired. □

Lemma 3.4. *Let $\langle\dot{\mathcal{U}}_n:n<\omega\rangle$ be a sequence of $\mathbb{P}$-names for open covers of a space $(\check{X},\tau^{\mathbb{P}})$, and $\langle\mathcal{H}_n:n<\omega\rangle$ a sequence of subsets of τ with the following properties.*

(1) For each $n<\omega$, $\mathcal{H}_n$ refines $\mathcal{V}_n(\dot{\mathcal{U}}_n)$.
(2) For all $x\in X$, for infinitely many $n<\omega$ there is $H\in\mathcal{H}_n$ such that $x\in H$.

Then there is a sequence $\langle\dot{\mathcal{W}}_n:n<\omega\rangle$ of $\mathbb{P}$-names with the following properties.

(1) For each $n<\omega$, $\Vdash_{\mathbb{P}}$ "$\dot{\mathcal{W}}_n\subseteq\check{\mathcal{H}}_n$ and $\dot{\mathcal{W}}_n$ refines $\dot{\mathcal{U}}_n$".
(2) $\Vdash_{\mathbb{P}}$ "$\bigcup_{n<\omega}\dot{\mathcal{W}}_n$ covers $(\check{X},\tau^{\mathbb{P}})$".

Proof. For each $n<\omega$, construct $\dot{\mathcal{W}}_n$ as in Lemma 3.3 from $\mathcal{H}_n$. We check that $\dot{\mathcal{W}}_n$'s meet the requirement (2). Fix $x\in X$ and $p\in\mathbb{P}$. Choose $m<\omega$ with $p\in P_m$. By the assumption, we can choose $n\geq m$ and $H\in\mathcal{H}_n$ with $x\in H$. Then, by Lemma 3.3, there is $r\in\mathbb{P}$ such that $r\leq p$ and $r\Vdash_{\mathbb{P}}$ "$\check{H}\in\dot{\mathcal{W}}_n$". This means that every $x\in X$ is forced to be covered by the union of $\dot{\mathcal{W}}_n$'s. □

Now we are going to prove the preservation of the Rothberger property. We use the equivalent conditions of the Rothberger property shown in the following lemma. The equivalence (1) ⇔ (2) is well-known, and (1) ⇔ (3) is easy.

Lemma 3.5. *For a topological space (X,τ), the following conditions are equivalent:*

(1) (the Rothberger property) For every sequence $\langle \mathcal{U}_n : n < \omega \rangle$ of open covers of (X, τ), there is a sequence $\langle U_n : n < \omega \rangle$ of open sets of (X, τ) such that
- *for all $n < \omega$, $U_n \in \mathcal{U}_n$, and*
- *for all $x \in X$ there is $n < \omega$ such that $x \in U_n$.*

(2) For every sequence $\langle \mathcal{U}_n : n < \omega \rangle$ of open covers of (X, τ), there is a sequence $\langle U_n : n < \omega \rangle$ of open sets of (X, τ) such that
- *for all $n < \omega$, $U_n \in \mathcal{U}_n$, and*
- *for all $x \in X$ there are infinitely many $n < \omega$ such that $x \in U_n$.*

(3) For every sequence $\langle \mathcal{U}_n : n < \omega \rangle$ of open covers of (X, τ), there is a sequence $\langle \mathcal{W}_n : n < \omega \rangle$ of sets of open sets of (X, τ) such that
- *for all $n < \omega$, $|\mathcal{W}_n| \leq 1$ (that is, $\mathcal{W}_n$ is either a singleton or $\emptyset$) and $\mathcal{W}_n$ refines $\mathcal{U}_n$, and*
- *for all $x \in X$ there is $n < \omega$ such that, there is $U \in \mathcal{W}_n$ such that $x \in U$.*

Theorem 3.6. *Suppose that (X, τ) is a topological space with the Rothberger property and $\mathbb{P}$ is a weakly endowed forcing notion. Then we have*

$$\Vdash_{\mathbb{P}} \text{“}(\check{X}, \tau^{\mathbb{P}}) \text{ has the Rothberger property”}.$$

Proof. Fix a sequence $\langle \dot{\mathcal{U}}_n : n < \omega \rangle$ of $\mathbb{P}$-names for open covers of $(\check{X}, \tau^{\mathbb{P}})$. Consider the sequence $\langle \mathcal{V}_n(\dot{\mathcal{U}}_n) : n < \omega \rangle$ of open covers of (X, τ). Using the condition (2) in Lemma 3.5, we can get a sequence $\langle U_n : n < \omega \rangle$ of open sets of (X, τ) such that

- for all $n < \omega$, $U_n \in \mathcal{V}_n(\dot{\mathcal{U}}_n)$, and
- for all $x \in X$ there are infinitely many $n < \omega$ such that $x \in U_n$.

Let $\mathcal{H}_n = \{U_n\}$ for each $n < \omega$. Now we apply Lemma 3.4 to $\langle \mathcal{H}_n : n < \omega \rangle$ to get a sequence $\langle \dot{\mathcal{W}}_n : n < \omega \rangle$. It is straightforward to check that $\langle \dot{\mathcal{W}}_n : n < \omega \rangle$ is forced to meet the condition (3) in Lemma 3.5. □

Scheepers pointed out that the preservation of the Menger property under forcing with a weakly endowed forcing notion is proved in the same way as the proof of Theorem 3.6. We will use Lemma 3.7 instead of Lemma 3.5.

Lemma 3.7. *For a topological space (X, τ), the following conditions are equivalent:*

(1) (the Menger property) For every sequence $\langle \mathcal{U}_n : n < \omega \rangle$ of open covers of (X, τ), there is a sequence $\langle \mathcal{H}_n : n < \omega \rangle$ of sets of open sets of (X, τ) such that
- *for all $n < \omega$, $\mathcal{H}_n$ is a finite subset of $\mathcal{U}_n$, and*
- *for all $x \in X$ there is $n < \omega$ such that, there is $U \in \mathcal{H}_n$ such that $x \in U$.*

(2) For every sequence $\langle \mathcal{U}_n : n < \omega \rangle$ of open covers of (X, τ), there is a sequence $\langle \mathcal{H}_n : n < \omega \rangle$ of sets of open sets of (X, τ) such that
- *for all $n < \omega$, $\mathcal{H}_n$ is a finite subset of $\mathcal{U}_n$, and*

- *for all $x \in X$ there are infinitely many $n < \omega$ such that, there is $U \in \mathcal{H}_n$ such that $x \in U$.*

(3) For every sequence $\langle \mathcal{U}_n : n < \omega \rangle$ of open covers of (X, τ), there is a sequence $\langle \mathcal{W}_n : n < \omega \rangle$ of sets of open sets of (X, τ) such that
- *for all $n < \omega$, $\mathcal{W}_n$ is finite and refines $\mathcal{U}_n$, and*
- *for all $x \in X$ there is $n < \omega$ such that, there is $U \in \mathcal{W}_n$ such that $x \in U$.*

Theorem 3.8. *Suppose that (X, τ) is a topological space with the Menger property and $\mathbb{P}$ is a weakly endowed forcing notion. Then we have*

$$\Vdash_{\mathbb{P}} \text{“}(\check{X}, \tau^{\mathbb{P}}) \text{ has the Menger property”}.$$

We turn to the preservation of selective screenability. We use the equivalent conditions of selective screenability shown in the following lemma, which is easy to check.

Lemma 3.9. *For a topological space (X, τ), the following conditions are equivalent:*

(1) (selective screenability) For every sequence $\langle \mathcal{U}_n : n < \omega \rangle$ of open covers of (X, τ), there is a sequence $\langle \mathcal{W}_n : n < \omega \rangle$ of sets of open sets of (X, τ) such that
- *for all $n < \omega$, $\mathcal{W}_n$ is pairwise disjoint and refines $\mathcal{U}_n$, and*
- *for all $x \in X$ there is $n < \omega$ and $U \in \mathcal{W}_n$ such that $x \in U$.*

(2) For every sequence $\langle \mathcal{U}_n : n < \omega \rangle$ of open covers of (X, τ), there is a sequence $\langle \mathcal{H}_n : n < \omega \rangle$ of sets of open sets of (X, τ) such that
- *for all $n < \omega$, $\mathcal{H}_n$ is pairwise disjoint and refines $\mathcal{U}_n$, and*
- *for each $x \in X$, there are infinitely many $n < \omega$ such that, there is $U \in \mathcal{H}_n$ with $x \in U$.*

Theorem 3.10. *Suppose that (X, τ) is a selectively screenable topological space and $\mathbb{P}$ is a weakly endowed forcing notion. Then we have*

$$\Vdash_{\mathbb{P}} \text{“}(\check{X}, \tau^{\mathbb{P}}) \text{ is selectively screenable”}.$$

Proof. Fix a sequence $\langle \dot{\mathcal{U}}_n : n < \omega \rangle$ of $\mathbb{P}$-names for open covers of $(\check{X}, \tau^{\mathbb{P}})$. Consider the sequence $\langle \mathcal{V}_n(\dot{\mathcal{U}}_n) : n < \omega \rangle$ of open covers of (X, τ). Using the condition (2) in Lemma 3.9, we can get a sequence $\langle \mathcal{H}_n : n < \omega \rangle$ of sets of open sets of (X, τ) such that

- for all $n < \omega$, $\mathcal{H}_n$ is pairwise disjoint and refines $\mathcal{V}_n(\mathcal{U}_n)$, and
- for each $x \in X$, there are infinitely $n < \omega$ such that, there is $U \in \mathcal{H}_n$ with $x \in U$.

Now we apply Lemma 3.4 to $\langle \mathcal{H}_n : n < \omega \rangle$ to get a sequence $\langle \dot{\mathcal{W}}_n : n < \omega \rangle$. It is straightforward to check that $\langle \dot{\mathcal{W}}_n : n < \omega \rangle$ is forced to meet the condition the condition (1) in Lemma 3.9. $\square$

4 Question

Although we defined the notion of weak endowments in a general fashion in Section 2, we do not have any examples of weakly endowed forcing notions other than $\mathbb{C}(\kappa)$ and $\mathbb{B}(\kappa)$ so far.

Question 4.1. *Are there any further examples of weakly endowed forcing notions?*

Appendix: Endowing Cohen forcing notions

Here we present a proof of Proposition 2.4, which is based on [3, Lemma 1.0].

Proof of Proposition 2.4. Fix an infinite cardinal κ and decompose $\mathbb{C}(\kappa)$ into an increasing union $\mathbb{C}(\kappa) = \bigcup_{n<\omega} C_n$ where $C_n = \{p \in \mathbb{C}(\kappa) : |p| \leq n\}$.

Fix $n < \omega$. We will claim the following:

> For every maximal antichain A in $\mathbb{C}(\kappa)$ there is a finite subset L of A such that, for every $p \in C_n$ there is $q \in L$ which is compatible with p.

Then we gather all L's, each corresponding to a maximal antichain, and it makes up the n-th weak endowment $\mathcal{L}_n$.

For $p \in \mathbb{C}(\kappa)$, $\operatorname{supp}(p)$ denotes the domain of p as a partial function from κ to 2, and for $F \subseteq \mathbb{C}(\kappa)$, let $\operatorname{supp}(F) = \bigcup\{\operatorname{supp}(p) : p \in F\}$.

Fix a maximal antichain A in $\mathbb{C}(\kappa)$. Pick any $a \in A$. Let $E_0 = \{a\}$ and $D_0 = \operatorname{supp}(E_0)$. For each $p \in \mathbb{C}(\kappa)$ with $\operatorname{supp}(p) \subseteq D_0$, choose exactly one $a_p \in A$ which is compatible with p. Let E_1 be the collection of such a_p's and $D_1 = \operatorname{supp}(E_0 \cup E_1)$. Similarly, for $i \leq n$, obtain E_i and D_i from D_{i-1}. Note that $D_i \supsetneq D_{i-1}$ holds for each i. Finally we let $L = \bigcup_{i \leq n} E_i$.

We check that this L works. Fix an arbitrary $p \in P_n$. Consider $n+1$ disjoint subsets $D_0, D_1 \setminus D_0, \ldots, D_n \setminus D_{n-1}$ of D_n. Since $|\operatorname{supp}(p)| \leq n$, $\operatorname{supp}(p)$ must be disjoint from some of those pieces, say $D_i \setminus D_{i-1}$ (let $D_{-1} = \emptyset$ for convention). Then there is $q \in E_i$ which is compatible with $p \restriction D_{i-1}$. Since $\operatorname{supp}(q) \subseteq D_i$ and $\operatorname{supp}(p) \cap (D_i \setminus D_{i-1}) = \emptyset$, q is compatible with p. □

Acknowledgement

I thank Marion Scheepers and Franklin D. Tall for many helpful comments on a provisional version of this article.

References

[1] D. Addis and J. Gresham, *A class of infinite-dimensional spaces Part I: Dimension theory and Alexandrof's problem*, Fund. Math. **101** (1978), 195–205.

[2] A. Dow, *Remote points in large products*, Topology Appl. **16** (1983), 11–17.

[3] A. Dow, F. D. Tall, and A. R. Weiss, *New proofs of the consistency of the normal Moore conjecture I*, Topology Appl. **37** (1990), 33–51.

[4] R. Grunberg, L. R. Junqueira, and F. D. Tall, *Forcing and normality*, Topology Appl. **84** (1998), 145–174.

[5] A. Iwasa, *Covering properties and Cohen forcing*, Topology Proc. **31** (2007), 553–559.

[6] K. Kunen, *Set theory: an introduction to independence proofs*, Studies in Logic, vol. 102, North Holland, 1980.

[7] M. Scheepers and F. D. Tall, *Lindelöf indestructivity, topological games and selection principles*, preprint.

endowment ◆◆◇◇◇ /ɪndaʊmənt/ (**endowments**)
[1] N-COUNT
An **endowment** is a gift of money that is made to an institution or community in order to provide it with an annual income.
[2] N-COUNT: usu with supp
If someone has an **endowment** of a particular quality or ability, they possess it naturally. [FORMAL]
[3] N-COUNT: usu N n
In finance, an **endowment** policy or mortgage is an insurance policy or mortgage which you pay towards each month and which should then provide you with enough money to pay for your house at the end of a fixed period. [BRIT]

—*COBUILD for Advanced Learner's English Dictionary New digital edition (Collins)*

en·dow·ment /endáʊmənt, ɪn-, ən- | ɪn-, en-, ən-/ 【初 15c; endow+-ment】
[名]
1a [U]（基金の）寄付（をすること）；遺贈.
b [C]（学校・病院などに寄付された）基金，寄付金；遺産.
2 [C]《正式》[通例 ～s]（生まれつきの）才能，資質.
3 [C]《豪》育児手当, 児童手当（child ～）.

—ジーニアス英和大辞典（大修館書店）